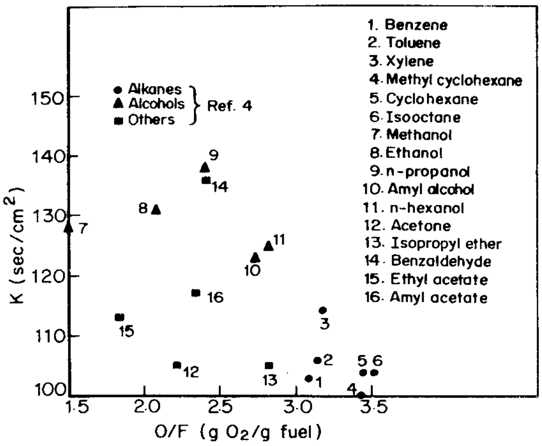

Fig. 1 Burning constant vs oxidizer/fuel ratio for liquid droplets.

where d_f is the flame diameter, $\mathscr{D}_f$, $\mathscr{D}_o$ are the diffusion coefficients of fuel and oxidizer, ρ_f, ρ_o are the vapor densities of fuel and oxidizer, p_o is the partial pressure of oxidizer in ambient air, and i is the stoichiometric ratio ($= O/F$).

Now the usual theory[3] of liquid droplet combustion including convective effects leads to

$$K \propto (1 - d/d_f) \tag{3}$$

Combining Eqs. (2) and (3), Essenhigh and Dreier get

$$K \propto \frac{O/F}{\alpha + O/F} \tag{4}$$

where $\alpha = \mathscr{D}_o p_o \rho_o / \mathscr{D}_f \rho_f$. Relation (4) has been used to justify the plot of K vs O/F. If we use the usual theory of liquid droplets for obtaining d_f/d, we get

$$\frac{d_f}{d} = \frac{\ln(1+B)}{\ln[1 + (p_o/i)(M_o/pM)]} \tag{5}$$

where B is the well-known transfer number, M_o and M are the molecular weights of oxidant and mixture, and p the total pressure. It can now be noticed that the more correct result (5) (only "more correct" since free convection and other effects have not been included in both cases) does not lead in any limit to Eq. (2). Also the exact result for K given by

$$K = \rho_l / 8\mathscr{D}\rho \ln(1+B) \tag{6}$$

where ρ_l = density of polymer, $\bar{\mathscr{D}}$ = molecular diffusivity, and ρ = density in the gas phase, does not produce Eq. (4) in any appropriate limit.

Suspecting that the correlation K vs O/F would be inappropriate, it was thought fit to check the correlation against known results on liquid droplets. It was particularly inviting to perform this since Essenhigh and Dreier also draw upon the conclusions from liquid droplet burning. The results of K vs O/F for a member of alkanes, alcohols, and others[4] have been plotted in Fig. 1. It is apparent that no trace of correlation can be seen. The only general conclusion appears to be that oxygen containing substances have larger K (smaller burning rate) than others.

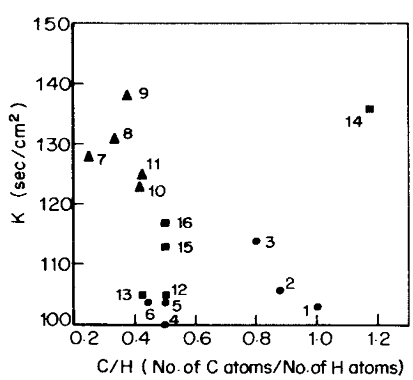

Fig. 2 Burning constant vs C/H ratio for liquid droplets.

Essenhigh and Dreier also attempt to invoke a correlation between K and C/H ratio on the basis that such a correlation exists for liquid droplets. Figure 2 shows the plot of K vs C/H for the same liquids cited in Fig. 1. Again the absence of any regular behavior is more than evident.

The liquid droplet theory by Godsave[3] worked out as long ago as 1953, is quite simple and contains the principal features of droplet combustion. The result for K as in Eq. (6) has been verified to be accurate by several investigators.[4] While the authors[2] use Godsave's theory to evolve Eq. (3), they use an approximate theory, neglecting the convection terms, to obtain Eq. (2) and consequently arrive at incorrect relations and conclusions.

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Transformation for the Numerical Integration of Systems of the Form $\ddot{X}(t) = F[X(t)]$

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MANY problems can be characterized by a second-order differential equation of the form

$$\ddot{X}(t) = F[X(t)] \tag{1}$$

where $X(t)$, $F[X(t)]$ are each n-vectors, and $F[X(t)]$ has a continuous partial derivative with respect to $X(t)$, for $X(t)$ in some subset of Euclidean n-space. Such differential equations may arise from Newton's second law of motion, or they may describe a two-point boundary-value problem. For numerical integration from t_0 to t_1 with known initial conditions $X(t_0)$, $\dot{X}(t_0)$, Eq. (1) is usually converted to the equivalent first-order system

$$\dot{X}(t) = Y(t) \tag{2}$$

$$\dot{Y}(t) = F[X(t)] \tag{3}$$

It is known[1] that the errors in the numerical integration of Eqs. (2) and (3) depend on the eigenvalues of the matrix

$$H(t) = \begin{bmatrix} \Theta & \vdots & I \\ \cdots & \cdots & \cdots \\ J[X(t)] & \vdots & \Theta \end{bmatrix} \tag{4}$$

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where Θ is an $n \times n$ null matrix, I is the $n \times n$ identity matrix, and $J[X(t)]$ is the $n \times n$ Jacobian matrix of $F[X(t)]$. $H(t)$ is the Jacobian matrix of the right-hand side of Eqs. (2) and (3). Typically, Eqs. (2) and (3) would be integrated from t_0 to t_1 [initial conditions $X(t_0)$ and $Y(t_0) \equiv \dot{X}(t_0)$ are known] with the single-step error not exceeding some predetermined bound. If Λ is defined to be

$$\Lambda = \max_t \max_j |\lambda_j[H(t)]| \tag{5}$$

where $\lambda_j[H(t)]$ is an eigenvalue of $H(t)$, large values of Λ will require a large number of integration steps to attain the requested accuracy. This can be costly in computer time, especially if Eqs. (2) and (3) must be repeatedly integrated, as would be the case in the shooting method solution of a two-point boundary value problem. Consequently, a method to reduce Λ is required. The following change of variables provides the desired effect.

Theorem

The change of variables

$$X(t) = Z(t) + (t-t_0)Y(t) \tag{6}$$

transforms the system $\dot{X}(t) = Y(t)$, $\dot{Y}(t) = F[X(t)]$ into

$$\dot{Z}(t) = -(t-t_0)F[X(t)] \tag{7}$$

$$\dot{Y}(t) = F[X(t)] \tag{8}$$

where $X(t)$ is given by Eq. (6). Furthermore, the Jacobian matrix of the system defined by Eqs. (7) and (8) has all eigenvalues equal to zero.

Proof

Since $\dot{X}(t) = Y(t)$, differentiation of Eq. (6) with respect to t yields $0 = \dot{Z}(t) + (t-t_0)\dot{Y}(t)$. But $\dot{Y}(t) = F[X(t)]$, hence Eqs. (7) and (8) hold. The Jacobian matrix of the system defined by Eqs. (7) and (8) is

$$M(t) = \begin{bmatrix} -(t-t_0)J[X(t)] & -(t-t_0)^2 J[X(t)] \\ J[X(t)] & (t-t_0)J[X(t)] \end{bmatrix} \tag{9}$$

where $J[X(t)]$ is the Jacobian matrix of $F[X(t)]$. But $M(t)$ can be expressed as

$$M(t) = [S(t)]^{-1} K(t) S(t) \tag{10}$$

where

$$S(t) = \begin{bmatrix} I & (t-t_0)I \\ \Theta & I \end{bmatrix} \tag{11}$$

$$K(t) = \begin{bmatrix} \Theta & \Theta \\ J[X(t)] & \Theta \end{bmatrix} \tag{12}$$

Therefore $M(t)$ is similar to $K(t)$, and clearly $K(t)$ has all eigenvalues equal to zero.

To illustrate the effectiveness of the transformation, Eq. (6), two example problems are given. Differential equations were integrated using a version of Gear's numerical integrator for nonstiff systems, Refs. 2 and 3. The requested single-step error in each case was 10^{-6}. All computations were done in double precision on the IBM 360/75 at the University of Waterloo.

Example 1[4]

The describing equations for a two-body central-force problem are

$$\dot{x}_1 = y_1 \qquad x_1(0) = 1.076 \tag{13}$$

$$\dot{x}_2 = y_2 \qquad x_2(0) = 0.0 \tag{14}$$

$$\dot{x}_3 = y_3 \qquad x_3(0) = 0.0 \tag{15}$$

$$\dot{y}_1 = -x_1/r^3 \qquad y_1(0) = 0.10165880 \tag{16}$$

$$\dot{y}_2 = -x_2/r^3 \qquad y_2(0) = 0.47228336 \tag{17}$$

$$\dot{y}_3 = -x_3/r^3 \qquad y_3(0) = 0.81801856 \tag{18}$$

$$r^2 = x_1{}^2 + x_2{}^2 + x_3{}^2 \tag{19}$$

$$0 \leqq t \leqq 2 \tag{20}$$

Applying the change of variables

$$x_j = z_j + t y_j \qquad (j = 1\text{–}3) \tag{21}$$

to the system, yields ($j = 1$–3)

$$\dot{z}_j = t x_j / r^3 \tag{22}$$

$$\dot{y}_j = -x_j / r^3 \tag{23}$$

with

$$z_j(0) = x_j(0) \tag{24}$$

The computational results are summarized in Table 1.

Table 1 Computational results for Example 1

	Untransformed system	Transformed system
Number of integration steps	40	26
$x_1(2)$	$3.5708780793 \times 10^{-7}$	$-3.9442372879 \times 10^{-7}$
$x_2(2)$	0.57600040220	0.57600026525
$x_3(2)$	0.99766180923	0.99766157203
$y_1(2)$	-0.88225044276	-0.88225101683
$y_2(2)$	$-1.9671679487 \times 10^{-2}$	$-1.9671740756 \times 10^{-2}$
$y_3(2)$	$-3.4072343131 \times 10^{-2}$	$-3.4072449252 \times 10^{-2}$

Example 2[5]

This example is a two-point boundary-value problem in which $\Lambda > 10^3$. The describing equations are

$$\dot{x}_1 = y_1 \qquad x_1(0) = 0.0 \tag{25}$$

$$\dot{x}_2 = y_2 \qquad x_2(0) = 0.0 \tag{26}$$

$$\dot{y}_1 = 10\sinh(10x_1) \qquad y_1(0) = B,\ \text{unknown} \tag{27}$$

$$\dot{y}_2 = 100x_2\cosh(10x_1) \qquad y_2(0) = 1.0 \tag{28}$$

$$0 \leqq t \leqq 1 \tag{29}$$

$$x_1(1) = 1.0 \tag{30}$$

The problem is to determine B so that Eq. (30) is satisfied. The variable $x_2(t) \equiv \partial x_1(t)/\partial B$. After making the change of variables

$$x_j = z_j + t y_j \qquad (j = 1, 2) \tag{31}$$

the system becomes

$$\dot{z}_1 = -10t\sinh(10x_1) \qquad z_1(0) = 0.0 \tag{32}$$

$$\dot{z}_2 = -100tx_2\cosh(10x_1) \qquad z_2(0) = 0.0 \tag{33}$$

$$\dot{y}_1 = 10\sinh(10x_1) \qquad y_1(0) = B \tag{34}$$

$$\dot{y}_2 = 100x_2\cosh(10x_1) \qquad y_2(0) = 1.0 \tag{35}$$

The shooting method[4,5] was used to solve the two-point boundary value problem in each case, with initial $B = 0$. The computational results are shown in Table 2.

Table 2 Computational results for Example 2

	Untransformed system	Transformed system
Iterations to convergence	7	7
Computing time	0.28 min	0.17 min
Number of integration steps at converged solution	344	130
$[x_1(1) - 1.0]$ at converged solution	3.464×10^{-8}	3.043×10^{-8}
Converged value of B	$3.5833049028 \times 10^{-4}$	$3.5833956485 \times 10^{-4}$

As can be seen from the two computational examples, numerical integration of the transformed system, Eqs. (7) and (8), is superior to numerical integration of the untransformed system, Eqs. (2) and (3), especially in the case where the Jacobian matrix of the original system has eigenvalues of large magnitude. Because the eigenvalues of the transformed Jacobian matrix are all zero, the method will also be effective if the original system is stiff.[6]

Similar transformations to reduce the magnitudes of the eigen values of the Jacobian matrix for a general first-order system may be found in Refs. 6 and 7.

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Additional Two-Dimensional Wake and Jet-Like Flows

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I. Introduction

A NUMBER of authors[1–4] have described wake and jetlike similarity solutions to the steady laminar, incompressible boundary-layer equations. The purpose of this Note is to extend these solutions into several new areas.

II. Equations

Solutions are sought to the two-dimensional incompressible boundary-layer equations

$$\partial u/\partial x + \partial v/\partial y = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + v\frac{\partial^2 u}{\partial y^2} \tag{2}$$

with the general boundary conditions

$$y = 0: \quad u = u_w(x),\ v = 0$$

$$y \to \infty: \quad u \to u_e(x)$$

A convenient form of the similarity equations may be obtained by assuming that

$$G'(\eta) = (u-u_e)/(u_w-u_e), \quad \eta = y/h(x) \tag{3}$$

so that Eqs. (1) and (2) become

$$\left.\begin{aligned} G''' + A[(G+B\eta)G'' - \beta(G'^2+2BG')] = 0 \\ G(0) = G'(\infty) = 0, \quad G'(0) = 1 \end{aligned}\right\} \tag{4}$$

where

$$A = \pm 1$$

$$B = u_e/(u_w - u_e)$$

$$u_e \sim u_w \sim x^{\beta/(2-\beta)}$$

$$h(x) = [(2-\beta)ABvx/u_e]^{1/2}$$

Wake and jetlike solutions to Eq. (4) have the added restriction that $G''(0) = 0$. The present work is intended to consider those values of A, B, and β for which such solutions can be found.

The parameter B essentially defines the wall to edge velocity ratio, and β is the usual velocity gradient parameter. The constant A is a more complicated parameter because it takes on the values of ± 1 depending on both B and β. If the solutions are to be real then h must be real and $(2-\beta)ABx/u_e \geq 0$. Furthermore, by considering the asymptotic behavior of Eq. (4) at large η, one may show that the outer boundary condition is approached through an exponential decrease (decay) of G' for the case when $AB > 0$ and $\beta \leq 2$. There are two other cases: first for $\beta > 2$, $AB > 0$ the outer boundary condition is again approached with exponential decay of G' but this is a so called "backward boundary layer"[5] in which $x/u_e < 0$. The second case is where $AB < 0$ with $\beta < -\frac{1}{2}$ and $x/u_e < 0$ and thus this is also a backward boundary layer but with asymptotic algebraic decay of G'. The latter case is not the usual kind of boundary layer although it has been discussed elsewhere[5,6] in some detail.

III. Discussion

Stewartson's[1] original work and the more recent paper by Kennedy[2] concerning the wakelike flows in the region $B < 0$ is shown in Fig. 1. Note that $B = -1$ and $\beta = -0.1988$ specifies the Falkner-Skan separation profile, and at $B = -1$ and $\beta = 0$ the solution corresponds to Chapman's free shear layer solution.[2] Steiger and Chen[3] have extended these results to $B > 0$ which are jet-like solutions (u_e may be considered positive without loss of generality; thus $B \geq 0$ corresponds to $u_e \leq u_w$ and these flows may be designated as jets). They did not point out however that the limiting case of $u_e = 0$ (i.e., $B = 0$) corresponds to Schlichting's[4] two-dimensional jet into a quiescent fluid. Steiger and Chen also reported the counterflow jet solutions where $-\frac{1}{2} \leq B \leq -\frac{1}{3}$ and $\beta \geq 1$. In a recent investigation,[7] no solutions with $G''(0) = 0$ were found in the $0 < \beta < 1$ range which confirms Steiger and Chen's observation.

Limit $\beta \to \pm\infty$

Steiger and Chen[3] reported that when $\beta = \infty$, $u_w/u_e = -2$ (i.e., $B = -\frac{1}{3}$), based on numerical integration of the similarity

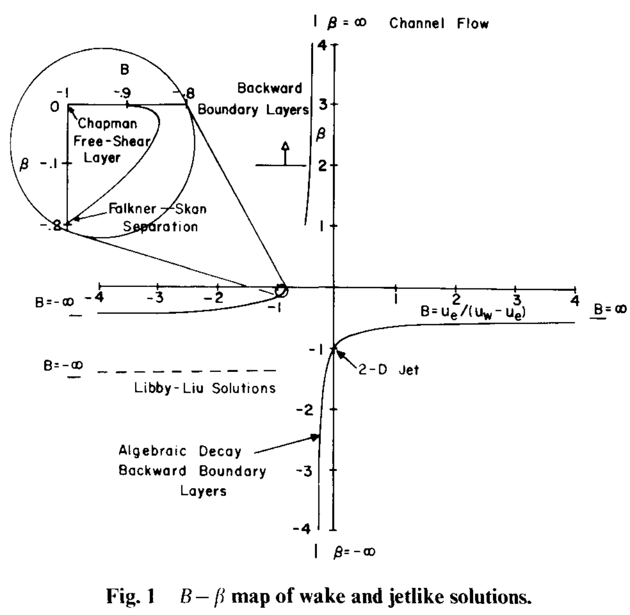

Fig. 1 $B-\beta$ map of wake and jetlike solutions.

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